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Nonlinear stability of the equilibria in a double-bar rotating system

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ABSTRACT

We study the nonlinear stability of the equilibria corresponding to the motion of a particle orbiting around a two finite orthogonal straight segment. The potential is a logarithmic function and may be considered as an approximation to the one generated by irregular celestial bodies. Using Arnold's theorem for non-definite quadratic forms we determine the nonlinear stability of the equilibria, for all values of the parameter of the problem. Moreover, the resonant cases are determined and the stability is investigated.

1. Introduction

The study of the existence and stability of stationary solutions is a central problem in the analysis of dynamical systems. In most cases, to determine the orbital stability is not a trivial matter. For some Hamiltonian systems, like the ones that can be found in celestial mechanics, the analysis of this stability can be done by taking into account only the quadratic terms in the Taylor expansion about the equilibrium point (see e.g. [1]). When this quadratic contribution is a definite form, the orbital stability is established as a consequence of the Lyapunov direct method. However, when the quadratic form is indefinite, a different method must be considered.

Arnold [2] presents a result to overcome this difficulty for Hamiltonian systems of two degrees of freedom. The hypothesis of this theorem requests a Hamiltonian expressed in its Birkhoff's normal form up to a certain degree. Thus, the complexity of the computations needed to bring the original Hamiltonian into this normalized form is the main difficulty in the use of Arnold's theorem. We present in this paper an application of this theorem for the study of the stability of stationary orbits around an uniformly rotating body composed by two orthogonal straight segments of the same length.

We consider the gravitational field created by two massive orthogonal straight segments rotating uniformly about an axis perpendicular to it. The main feature of an irregularly shaped celestial body, like many asteroids, with a significant effect on the orbits around such bodies is the irregular shape (see [3,4]). For this particular body, we can express the potential function in closed form that will allow us to carry out the analysis of the stability of the equilibria, in a synodic frame.

Asteroids and planetary satellites belong to the class of natural irregular bodies that are in pure rotation. They are old objects in the solar system and have reached the state of lowest energy for a given angular momentum, i.e., a pure rotation

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about the principal axis of highest moment of inertia; any primeval nutation faded away because nutation induces time-varying internal stresses that dissipate mechanical energy through hysteresis cycles (see [5,6]).

Concerning the Lyapunov stability of the systems, Arnold's and Markeev's theorems only work for systems with two degrees of freedom and in our model these hypothesis are fulfilled since our systems are planar.

The paper is organized as follows. Firstly in Section 2, we formulate the problem and obtain the equilibria. Their linear stability is analyzed in Section 3. The nonlinear stability of those points linear stable is determined using Amold's theorem in Section 4, and finally in Section 5, the stability of the resonant case is determined using as a key tool a result of Markeev [7]. We remark that the algorithms for obtaining normal forms were obtained by using a commercial symbolic package (Mathematica).

2. Equations of motion and equilibria

Since in our model we approximate the irregular body by two straight orthogonal segments, we shall assume that the segment uniformly rotates with angular velocity ω about the z-axis (perpendicular to the segment and fixed in the space). With this, we define a synodic reference frame $Oxyz$, with origin at the center of mass O, and such that the segment lies on the axis Ox.

Let us consider a straight segment of length 2ℓ and mass M. Assuming the linear mass density to be constant, the gravitational potential per unit mass created by this one dimensional body at a certain point P in the space may be expressed in closed form as

$$U(P) = -\frac{GM}{2\ell}\log\left(\frac{r_1+r_2+2\ell}{r_1+r_2-2\ell}\right), \tag{1}$$

equation that depends only on the distances: the length of the segment 2ℓ, and the distances r_1 and r_2 of the particle to the end points of the segment.

The number of possible parameters is reduced to only one by an appropriate scaling of the Hamiltonian (see [8]). In our case, after scaling length and time in such a way that 2ℓ is the unit of length and $1/\omega$ the unit of time, the Hamiltonian corresponding to the motion of a particle moving on the xy-plane is the function

$$\mathcal{H}(x,y,X,Y) = \frac{1}{2}(X^2+Y^2) - (xY - yX) - k\mathcal{V} \tag{2}$$

with

$$\mathcal{V}(x,y) = k\left(\log\left(\frac{r_1+r_2+1}{r_1+r_2-1}\right) + \log\left(\frac{r_3+r_4+1}{r_3+r_4-1}\right)\right)$$

being $r_1 = \sqrt{(x-\frac{1}{2})^2+y^2}$, $r_2 = \sqrt{(x+\frac{1}{2})^2+y^2}$, $r_3 = \sqrt{x^2+(y-\frac{1}{2})^2}$, $r_4 = \sqrt{x^2+(y+\frac{1}{2})^2}$ and $k = GM/(\omega^2(2\ell)^3) \in (0,\infty)$ is a dimensionless parameter, that represents the ratio of the gravitational acceleration to centrifugal acceleration. $k < 1$ means fast rotation of the segment, whereas $k > 1$ means slow rotation.

The equations of motion are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial X}, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial Y}, \qquad \frac{\mathrm{d}X}{\mathrm{d}t} = -\frac{\partial\mathcal{H}}{\partial x}, \qquad \frac{\mathrm{d}Y}{\mathrm{d}t} = -\frac{\partial\mathcal{H}}{\partial y}. \tag{3}$$

The equilibria will be the solutions of the system obtained by making zero the right-hand members of these equations. It is easy to prove that there are no solutions with x and y simultaneously non-zero. Therefore, other possibilities are, after replacing the values of $X = -y$ and $Y = x$ from the first two equations in the two remaining, either $x = 0, y = 0$ or $x = \pm y$.

Equilibria on the x-axis and y-axis, that will be denoted *collinear* points, are those that satisfy the conditions $x = 0$ or $y = 0$. Let x_0 be the x-coordinate of this equilibrium; because of the symmetry with respect to the y-axis, we can assume that the point is located to the right of the segment ($x_0 > 1/2$). After a numerical analysis of the equations (3) is obtained that it has a unique root in the interval $(1/2,\infty)$, and moreover, the root lies outside of the segment. Hence, by symmetry, there are two collinear equilibria, symmetric to each other with respect to the origin (Fig. 1).

One could think in obtaining another equilibrium point inside the segment, albeit it has no physical meaning. In that case, if x_0 is the coordinate of this point, $0 \le x_0 < 1/2$. But actually this solution is spurious, since inside the segment $s = r_1 + r_2 = 1$, which is a singularity of the logarithmic function. Consequently, we conclude that the two equilibria obtained the only ones on the x-axis, that we dubbed $E_1(x_0 > 0)$ and $E_2(-x_0 < 0)$, symmetrical each other with respect to the y-axis. By symmetry, in the y-axis are two equilibria denoted by $E_3(y_0 > 0)$ and $E_4(y_0 < 0)$.

Equilibria when $x = y$ will be denoted *bisector* equilibria. With this, the condition for the existence of this equilibria are

$$k = f(y) = \frac{y\left(65y^4 + 8y^2\sqrt{1+64y^4} - \sqrt{1+64y^4} + 1\right)}{4\left(\sqrt{8y^2-4y+1} - \sqrt{8y^2+4y+1} + 4y\left(\sqrt{8y^2+4y+1} + \sqrt{8y^2-4y+1}\right)\right)}.$$

After an analysis of the previous equation, the number of bisector equilibria are four and denoted by $C_i(y_0 > 0)$ with $i = 1\ldots4$.

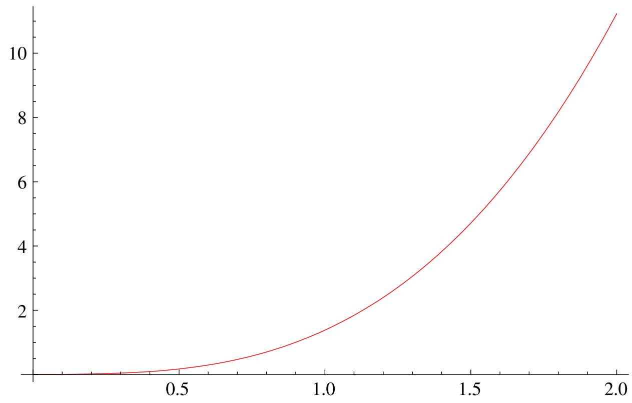

Fig. 1. Graphical representation of the function $f(y)$. In the y-axis are represented the parameter k.

3. Linear stability of the equilibria

Due to the symmetries of the problem, the stability of four bisector points is the same, as well as for the four collinear points, therefore it will suffice to study the stability of one of each type, namely, E_i and C_i.

By translating the origin of coordinates to one of the equilibria by a canonical transformation and by computing the Taylor expansion, we get $\mathcal{H} = \sum_{j\geq 0} \mathcal{H}_j$, where each term $\mathcal{H}_j$ is an homogeneous polynomial of degree j in the new variables $x = (\xi, \eta, P_\xi, P_\eta)$. Since it is an expansion in the vicinity of an equilibrium, $\mathcal{H}_0$ is a constant (the value of the Hamiltonian in the equilibrium point), whereas $\mathcal{H}_1$ is null because it is the gradient of $\mathcal{H}$ evaluated at the equilibrium. The linear stability of the equilibrium points is determined from the variational equations derived from the quadratic term $\mathcal{H}_2$, concretely from the system $\frac{\mathrm{dx}}{\mathrm{dt}} = \mathcal{J}\mathcal{A}x$, with $\mathcal{H}_2 = \frac{1}{2}x^T\mathcal{A}x$, where $\mathcal{J}$ is the standard 4×4 symplectic matrix and $\mathcal{A}$ a real symmetric matrix associated to the quadratic part of the Taylor expansion. At the equilibria, this matrix takes the values

$$\mathcal{J}\mathcal{A}(E_1) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -a_1 & 0 & 0 & 1 \\ 0 & -b_1 & -1 & 0 \end{pmatrix}$$

for the point E_1, and

$$\mathcal{J}\mathcal{A}(C_1) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -a_2 & -b_2 & 0 & 1 \\ -b_2 & -a_2 & -1 & 0 \end{pmatrix}$$

for C_1, where the coefficients a_i and b_i with $i = 1, 2$ functions of the parameter y.

The characteristic polynomial of the matrix $\mathcal{J}\mathcal{A}(E_1)$ is the bi-quadratic polynomial

$$P_{E_1}(\lambda) = \lambda^4 + (a_1 + b_1 + 2)\lambda^2 + (b-1)(a-1).$$

An numerical analysis of the coefficients $P_{E_1}(\lambda)$ shows that as one eigenvalue with positive real part, which means that the collinear points are always linearly unstable, hence Lyapunov unstable.

For the bisector points, the characteristic polynomial of $\mathcal{J}\mathcal{A}(C_1)$ is

$$P_{C_1}(\lambda) = \lambda^4 + 2(a_2 + 1)\lambda^2 + (b_2^2 + (a_2 - 1)^2).$$

The bisector equilibrium is linearly stable if $(a_2 + 1) > 0$ and

$$discrim(P_{C_1}(\lambda)) > 0.$$

Plot the functions $a_2 + 1$ and $discrim(P_{C_1})$ as functions of the variable y we observe that the bisector equilibrium are linearly stable in the interval $(y_c, +\infty)$ with $y_c = 0.793903722269259$.

For the limit value y_c, the eigenvalues are multiple purely imaginary, but for this value, the matrix $\mathcal{J}\mathcal{A}(E_2)(\delta_c)$ is non-diagonalizable and, consequently, the solution is unstable.

In conclusion, we determined that collinear points are everywhere unstable, and that bisector points are unstable for $y \in (0, y_c]$ and linearly stable for $y \in (y_c, +\infty)$ (Fig. 2).

In order to study the orbital stability of the bisector points we must take into consideration terms beyond the quadratic part in the Taylor expansion of the Hamiltonian about the equilibrium point.

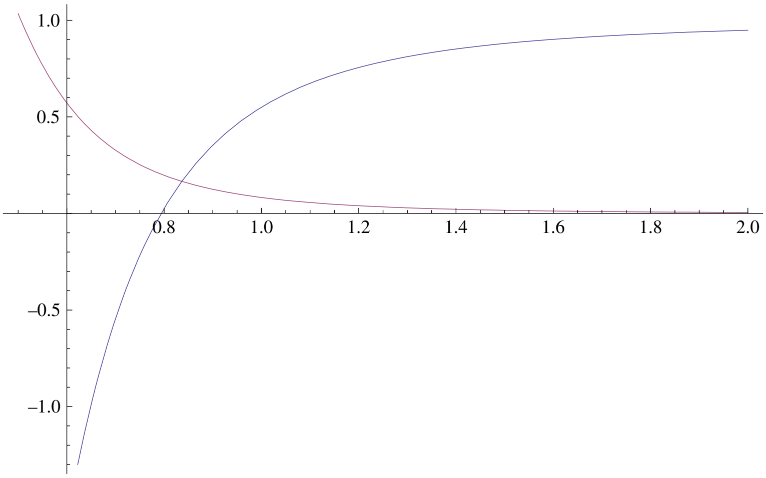

Fig. 2. The graphical representation of $discrim(P_{C_1}(\lambda))$ is the blue curve and the red curve is the graphical representation of $(a_2 + 1)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

4. Nonlinear stability of the equilibria

If the quadratic part $\mathcal{H}_2$ of the Hamiltonian were a definite form, Lyapunov's stability theorem would confirm that the equilibria is orbital stable (see [9]). Unfortunately, as we shall see, in our case the quadratic form is not sign definite and, a simple appeal to Lyapunov's stability theorem cannot be made. However, for that case stability can be investigated via Arnold's theorem (see [2]) that determines the stability of the two degrees of freedom system under certain conditions. Amold's theorem was already used in [10–12,7] to study the equilibria of the planar restricted problem of three bodies (RTBP). Later on, Meyer and Schmidt (1986) gave a new proof and reformulated the theorem. Deprit and López-Moratalla [13] applied it to the problem of the stationary satellites and automatized the normalization of the Hamiltonian facilitating the use of this result. More recently, it has been applied to some generalizations of the RTBP (see [14] or [15]). The theorem, as formulated by Meyer and Schmidt [16] has the following form.

Theorem 1 (*Arnold*)**.** *Consider a Hamiltonian $\mathcal{H}$ expressed, in the action and angle variables $(\psi_1, \psi_2, \Psi_1, \Psi_2)$, as*

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_4 + \cdots + \mathcal{H}_{2n} + \tilde{\mathcal{H}},$$

where

- *$\mathcal{H}$ is real analytic in a neighborhood of the origin R^4,*
- *$\mathcal{H}_{2k},\ 1 \le k \le n$, is a homogeneous polynomial of degree k in Ψ_j, with real coefficients. In particular,*

$$\mathcal{H}_2 = \omega_1\Psi_1 - \omega_2\Psi_2, \quad 0 < \omega_1,\ 0 < \omega_2,$$
$$\mathcal{H}_4 = \frac{1}{2}(A\Psi_1^2 - 2B\Psi_1\Psi_2 + C\Psi_2^2).$$

- *$\tilde{\mathcal{H}}$ has a power expansion in Ψ_j which starts with terms at least of order $n + 1/2$.*

Under this assumptions, the origin is a stable equilibrium provided for some $k,\ 2 \le k \le n,\ \mathcal{H}_2$ does not divide $\mathcal{H}_{2k}$ or equivalently, provided $D_{2k} = \mathcal{H}_{2k}(\omega_2,\ \omega_1) \neq 0$ and for $2 \le j < k,\ D_{2j} = \mathcal{H}_{2j}(\omega_2,\ \omega_1) = 0$ (Fig. 3).

The second condition implies that $\mathcal{H}$ is in Birkhoff's normal form up to terms of degree $2n$ and hence some non-resonance assumption on ω_j. (There is no smallness condition on the parameters in the statement of Arnold's theorem).

Only the bisector points for values of the parameter $y \in (y_c, +\infty)$ need to be considered because they are the only points that enjoy linear stability. The different terms of the Taylor expansion are

$$\mathcal{H}_2 = \frac{1}{2}(P_\xi^2 + P_\eta^2) - (\xi P_\eta - \eta P_\xi) + \frac{1}{2}\left(\frac{a_2}{2}\xi^2 + \frac{a_2}{2}\eta^2 + b_2\xi\eta\right), \tag{4}$$

for the quadratic part, and

$$\mathcal{H}_n = \sum_{i+j=n} \alpha_{ij}\xi^j\eta^j, \quad n > 2, \tag{5}$$

for the remaining terms, the coefficients α_{ij} may be expressed as functions of the different (but equivalent) parameters of the problem.

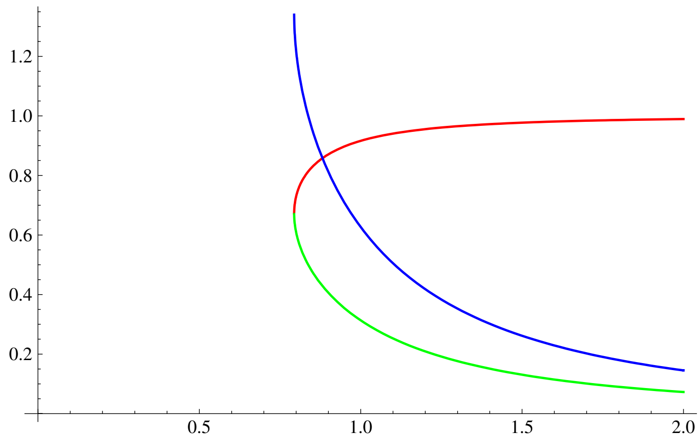

Fig. 3. Graphics of the frequencies ω_1 and ω_2 in function of the parameter y. The blue curve is the representation of $2\omega_1$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Let $\mathcal{A}$ the matrix associated to the real quadratic form $\mathcal{H}_2$, and 1et $\pm$ iω_1, $\pm$iω_2 the eigenvalues of the matrix $\mathcal{J}\mathcal{A}$, with $\omega_j \in \mathbb{R}$, and $\omega_1 > \omega_2$. It is possible to define a symplectic transformation such that the quadratic part (5) is converted into

$$\mathcal{H}_2 = \frac{\omega_1}{2}(P_1^2 + q_1^2) - \frac{\omega_2}{2}(P_2^2 + q_2^2), \tag{6}$$

that is to say, into the subtraction of two harmonic oscillators with frequencies ω_1 and ω_2. This frequencies, depending of the parameter y are represented in the following graphic.

Having $\mathcal{H}_2$ as required for Arnold's theorem is simple. It is achieved with the help of the Poincaré transformation

$$q_j = \sqrt{2\Psi_i}\sin\psi_j, \qquad P_j = \sqrt{2\Psi_i}\cos\psi_j, \quad (i = 1, 2) \tag{7}$$

the Hamiltonian (6) has the form in action-angle variables

$$\mathcal{H}_2 = \omega_1\Psi_1 - \omega_2\Psi_2. \tag{8}$$

Another requirement of the Arnold theorem is to have the Hamiltonian in normal form up to a certain order. At this point, let us recall that a given Hamiltonian

$$\mathcal{H} = \sum_{i\geq 2}\mathcal{H}_j$$

is, in normal form, up to a degree k, if Poisson's brackets $(\mathcal{H}_2;\ \mathcal{H}_j) = 0,\ \forall j \leq k$, or equivalently, when $\mathcal{H}_j \in \ker L_2$ where $L_2(F) = (\mathcal{H}_2; F)$ is the Lie derivative associated with $\mathcal{H}_2$.

We performed the normalization by means of the Lie–Deprit method (see [11]). Instead of using action and angle variables (7) we used the set of complex variables (so called Birkhoff variables),

$$\begin{aligned} u &= \sqrt{\Psi_1}\mathrm{e}^{\mathrm{i}\psi_1}, \qquad U = -\mathrm{i}\sqrt{\Psi_1}\mathrm{e}^{-\mathrm{i}\psi_1}, \\ v &= \sqrt{\Psi_2}\mathrm{e}^{-\mathrm{i}\psi_2}, \qquad V = \mathrm{i}\sqrt{\Psi_2}\mathrm{e}^{\mathrm{i}\psi_2}, \end{aligned} \tag{9}$$

since handling polynomials requires less computer requirements than trigonometric functions.

In these complex variables, the first term $\mathcal{H}_2$ is

$$\mathcal{H}_2 = \mathrm{i}\omega_1 uU + \mathrm{i}\omega_2 vV, \tag{10}$$

and the Lie derivative, L_2, expressed in this set of variables is

$$L_2(F) = (\mathcal{H}_2; F) = \mathrm{i}\omega_1\left(U\frac{\partial F}{\partial U} - u\frac{\partial F}{\partial u}\right) + \mathrm{i}\omega_2\left(V\frac{\partial F}{\partial V} - v\frac{\partial F}{\partial v}\right).$$

Applied to a generic monomial in the variables (u, v, U, V) is

$$L_2(u^mU^nv^pV^q) = (\mathrm{i}\omega_1(n-m) + \mathrm{i}\omega_2(q-p))u^mU^nv^pV^q,$$

hence, the elements of the kernel of the Lie derivative can be now easily identified by the equivalence

$$u^mU^nv^pV^q \in \ker L_2 \Leftrightarrow \omega_1(n-m) + \omega_2(q-p) = 0.$$

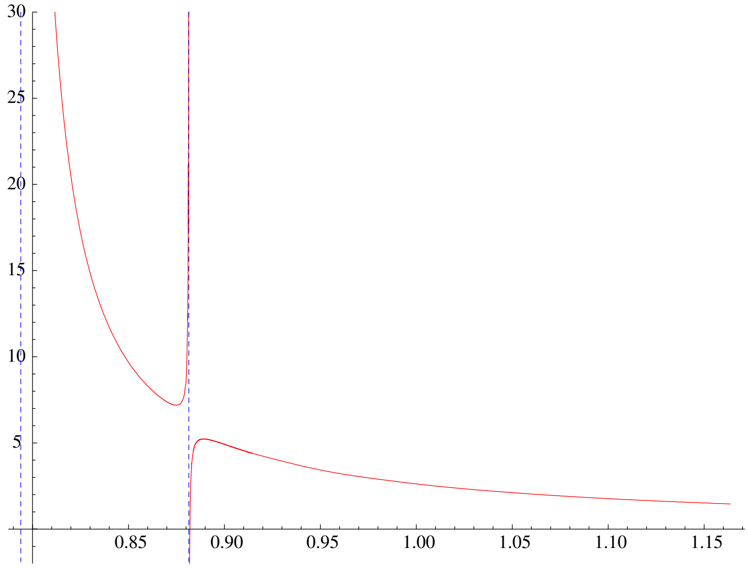

Fig. 4. Graphics representation of $D_4(y)$. The two vertical asymptotes reflect the 1:1 resonance and 2:1 resonances.

We analytically performed the normalization up to order 4 carrying the variable y as parameter along the computations, obtaining

$$\mathcal{H}_2 = \mathrm{i}\omega_1 u' U' + \mathrm{i}\omega_2 v' V', \qquad \mathcal{H}_3 = 0,$$
$$\mathcal{H}_4 = -Au'^2 U'^2 - 2Bu'U'v'V' - Cv'^2 V'^2,$$

with the coefficients of $\mathcal{H}_4$ functions of y.

This normalized Hamiltonian, written again in action-angle variables is

$$\mathcal{H} = (\omega_1 \Psi_1 - \omega_2 \Psi_2) + (A\Psi_1^2 - 2B\Psi_1\Psi_2 + C\Psi_2^2) + \cdots,$$

(we dropped the primes for the sake of simplifying the notation).

According to Arnold's theorem, stability is ensured if the function

$$D_4 = \mathcal{H}_4(\omega_2, \omega_1) = A\omega_2^2 - 2B\omega_1\omega_2 + C\omega_1^2,$$

is non-zero. The frequencies ω_1 and ω_2 are functions of the parameter y, thus, we can be obtained D_4 completely in terms of y but is very cumbersome to put analytically. A graphic analysis of the function $D_4(y)$ is carried out.

Since we are interested only in the interval $(y_c, +\infty)$, we find that in such an interval (see Fig. 4), D_4 has a zero at $y_0 \approx 0.882005135282627$ and two essential singularities, one at $y_1 = 0.881439850672093$, and another one at the end of the interval y_c, but we showed that for this value, which corresponds to the 1:1 resonance, the equilibria are unstable.

In conclusion, we can affirm that the bisector points are stable for all values of the y in the open interval $(y_c, +\infty)$ except, perhaps, for the points y_0 and y_1, that must be analyzed separately.

The value y_1 (vertical asymptote) corresponds to the resonance 2:1; Arnold's theorem is not applicable and this case will be studied in the next section.

In as much as y_0 is a zero of D_4, we need to push the normalization until order six and see whether the discriminant $D_6(y_0)$ is null or not. Since now y is a real number and not a generic parameter, we perform the normalization by using floating point arithmetics, which speeds up the normalization algorithm. The numerical values of the $D_4(y_0) = 6.5 \times 10^{-15}$ and $D_6(y_0) = 4.0567$ which, with an accuracy of 10^{-15}, ensures that the bisector problem is also stable for y_0.

So far, by using Arnold's theorem, we have determined the orbital stability of the bisector points for all the values of the interval $(y_c, +\infty)$ except for the resonant case y_1.

5. Stability of the resonant case

5.1. The 2:1 resonance

When $\delta = y_1$, the frequencies are

$$\omega_1 = 0.8579369494531961529,$$

and

$$\omega_2 = 0.4289684747350657406,$$

that is to say, $\omega_1 = 2\omega_2$, thus, we are dealing with the resonance 2:1, and the resonant cases are excluded from Arnold's theorem. However, Markeev [7] gave some results about the nonlinear stability precisely for resonances. For the third order resonance, the result given by Markeev says the following.

Theorem 2 (*Markeev*). *If the real normal form of the Hamiltonian contains resonant terms in the form*

$$\mathcal{H} = 2\omega\Psi_1 - \omega\Psi_2 - \sqrt{\omega(A_{12}^2 + B_{12}^2)}\Psi_2\sqrt{\Psi_1}\cos(\psi_1 + 2\psi_2) + \mathcal{O}(\Psi_1^2 + \Psi_2^2),$$

if $A_{12}^2 + B_{12}^2 \neq 0$, *then the equilibrium position is unstable.*

The normalized Hamiltonian are

$$\mathcal{H} = 2\omega\Psi_1 - \omega\Psi_2 - \Psi_2\sqrt{\Psi_1}\left(\frac{5}{1000}\cos(\psi_1 + 2\psi_2) + \frac{50}{1000}\sin(\psi_1 + 2\psi_2)\right)$$

which is precisely in the form of Markeev's theorem after a simply trigonometric transformation. Consequently, we conclude that, for the resonant case, the bisector equilibrium is unstable.

6. Conclusions

As a conclusions we state that the gravity field of two finite orthogonal straight segment can be expressed in closed form as a logarithmic function and is used to model the potential of irregular celestial bodies. Using as a key tool Arnold's theorem for non-definite quadratic forms we determine the orbital stability of the equilibria for all values of the parameters of the problem, resonant cases included.

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